Introduction to Topology
拓撲學簡介

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Introduction to Topology

outline · outcomes · prerequisites

Outline

- Topology
  - *What the hell is it?*
  - *Chaotic, discrete, and standard topology*
  - *Upgrading old definitions*
- Topological space
- Continuity
- Homeomorphism
  - *What do I mean by this crazy word?*
  - *A coffee mug is a donut.*
  - *Luckily, we can draw a map of Hong Kong.*

Outcomes

- Understand basic (but rigorous) notations and terminologies of topology
- Even the odds of browsing Wiki
- Have fun

Prerequisites

- High-school set theory
- MATH1510

Topology
What the hell is it?

Why topology?

Allows defining continuity!

Topology

What the hell is it?

Any set M

Its power set $\mathscr{P}(M)$

Topology axioms

A set $\mathcal{O} \subset \mathscr{P}(M)$ is a topology of M, iff:

- $\emptyset \in \mathcal{O}, M \in \mathcal{O}$;
- For any $U \in \mathcal{O}$ and $V \in \mathcal{O}$, we have $U \cap V \in \mathcal{O}$; （对有限个 intersection 封闭）
- For any $U_\alpha \in \mathcal{O}$, $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{O}$.（对有限与无限个 union 封闭）

Topology

chaotic · discrete · standard

- 对任意一个集合 M，很容易找出它的两个 topology：
  - chaotic topology: $\mathscr{O} = \{\emptyset, M\}$.
  - discrete topology: $\mathscr{O} = \mathscr{P}(M)$.

Topology

chaotic · discrete · standard

现在介绍 Standard topology （$\mathbb{R}^k$ 特有）

请问，在 Euclidean metric 下，$\mathbb{R}^k$ 上的开集是怎么定义的？

「其中每个点都是内点 (interior point) 的集合。」

定义 standard topology $\mathcal{O}_{std}$ ：Euclidean metric范畴内，全体开集的集合。

⚠️ 我们一会要重新定义开集！现在的定义是初等的！

Topology
upgrading old definitions

现在重新定义开集！

Suppose $\mathscr{O}$ is a topology of M. A subset $U \subset M$ is open (w.r.t. $\mathscr{O}$), iff

$$U \in \mathscr{O}.$$

By def., topology is a collection of open subsets of set M.

比用 metric 定义出来的开集更广泛。

Topology

upgrading old definitions

现在重新定义闭集！

$$U \text{ is closed iff } U^c \text{ is open.}$$

还可以继续定义 neighbourhood, limit point, compactness（紧致性）, ...

自行Wiki，已经没有阅读障碍了。

Topological Space

给集合 M 指定一个 topology $\mathcal{O}$，则形成 topological space $(M, \mathcal{O})$。

空间 = 集合 + 某种/某些结构

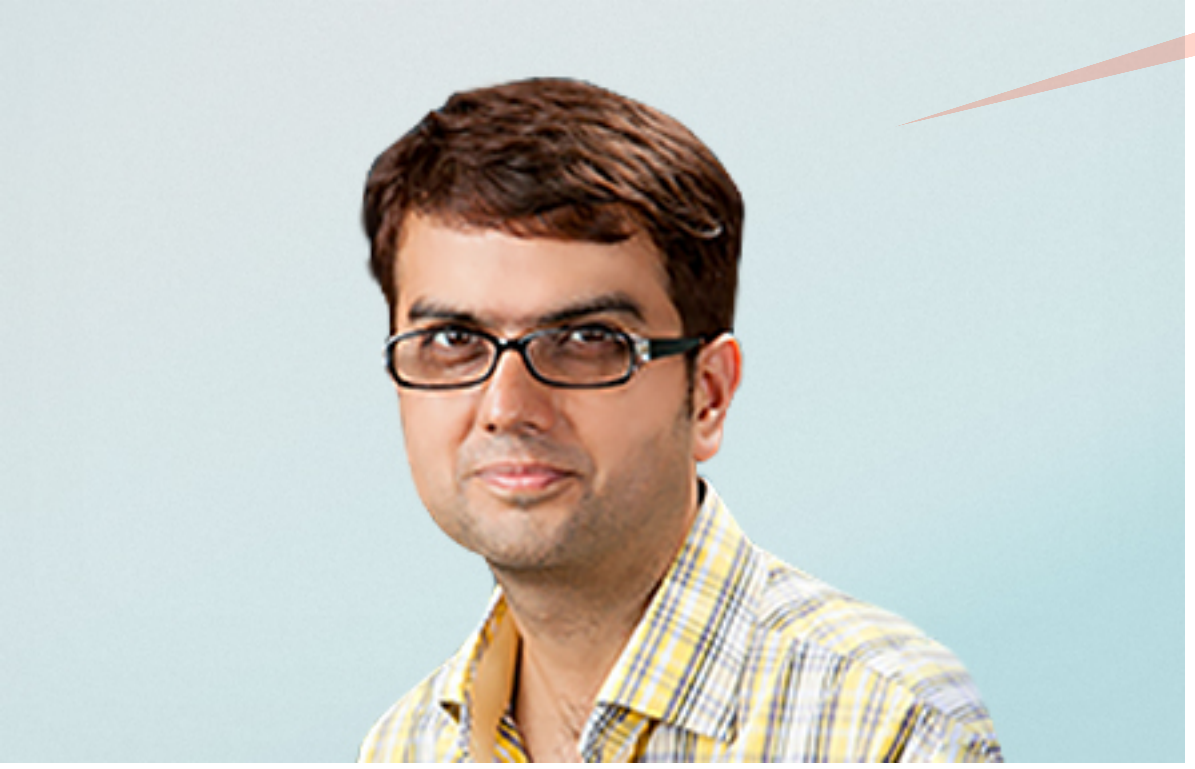
Continuity

考虑一个 map $x : U \to V$，希望定义 map 的连续性。

MATH1510：提到了 $x : \mathbb{R} \to \mathbb{R}$ 的连续性。

ESTR1005：定义了 $x : \mathbb{R} \to \mathbb{R}$ 的连续性。

炸鸡曰：「$\forall \varepsilon > 0, \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$。」



Continuity

拓扑空间是能定义连续性的最基本的空间。

首先介绍一个记号：$\mathrm{preim}_f(V) = \{u \mid f(u) \in V\}$.（原像/preimage）

Consider two topological spaces $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$. A map $f : M \to N$ is said to be continuous (w.r.t. $\mathcal{O}_M$ and $\mathcal{O}_N$), iff:

$$\forall V \in \mathcal{O}_N, \mathrm{preim}_f(V) \in \mathcal{O}_M.$$

「开集的原像是开集。」

Continuity

simple but makes sense （拓扑定义能否符合初等的几何直觉？）

fact: 当情况退化到 $\mathbb{R} \to \mathbb{R}$ 并采用 $\mathcal{O}_{std}$ 时，topological continuity 等价于：函数处处满足 Jaggi continuity。

先证 topo implies Jaggi. 对任意 $x_0 \in X$，取任意半宽为 ε 的开区间 $E = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$。它显然是 std topo 规定的开集。据 topo cont.,
$D \triangleq \text{preim}_f(E)$ 是开集。显然有 $x_0 \in D$。据开集之定义， 这个点是内点，即$\exists \delta : (x_0 - \delta, x_0 + \delta) \subset D$。
于是， $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$。此即 Jaggi cont.

再证 Jaggi implies topo. 反证法。任取开集 $E \subset \mathbb{R}$。若 $D \triangleq \text{preim}_f(E)$ 不是开集，则在 D 中存在 x，使得 $\forall \delta : (x - \delta, x + \delta) \not\subset D$。 但由于 E 开，$f(x)$ 一定是 E 的内点，即存在 $P \triangleq (f(x) - \varepsilon, f(x) + \varepsilon) \subset E$。据 Jaggi cont，
$\exists \delta : (x - \delta, x + \delta) \subset \text{preim}_f(P) \subset \text{preim}_f(E) = D$
。矛盾！

Continuity

若 f 和 g 连续，则 $g \circ f$ 连续

(easy to show)

Homeomorphism

What do I mean by this crazy word?

描绘两个拓扑空间在连续变换意义下的 equivalence relation。

Two topological spaces $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ are said to be homeomorphic, iff there exists a map $f : M \to N$, such that:

- f is bijective;
- both f and f^{-1} are continuous.

Notation: $(M, \mathcal{O}_M) \simeq (N, \mathcal{O}_N)$.

容易验证此定义满足 equivalence relation 的三个公理。

Homeomorphism

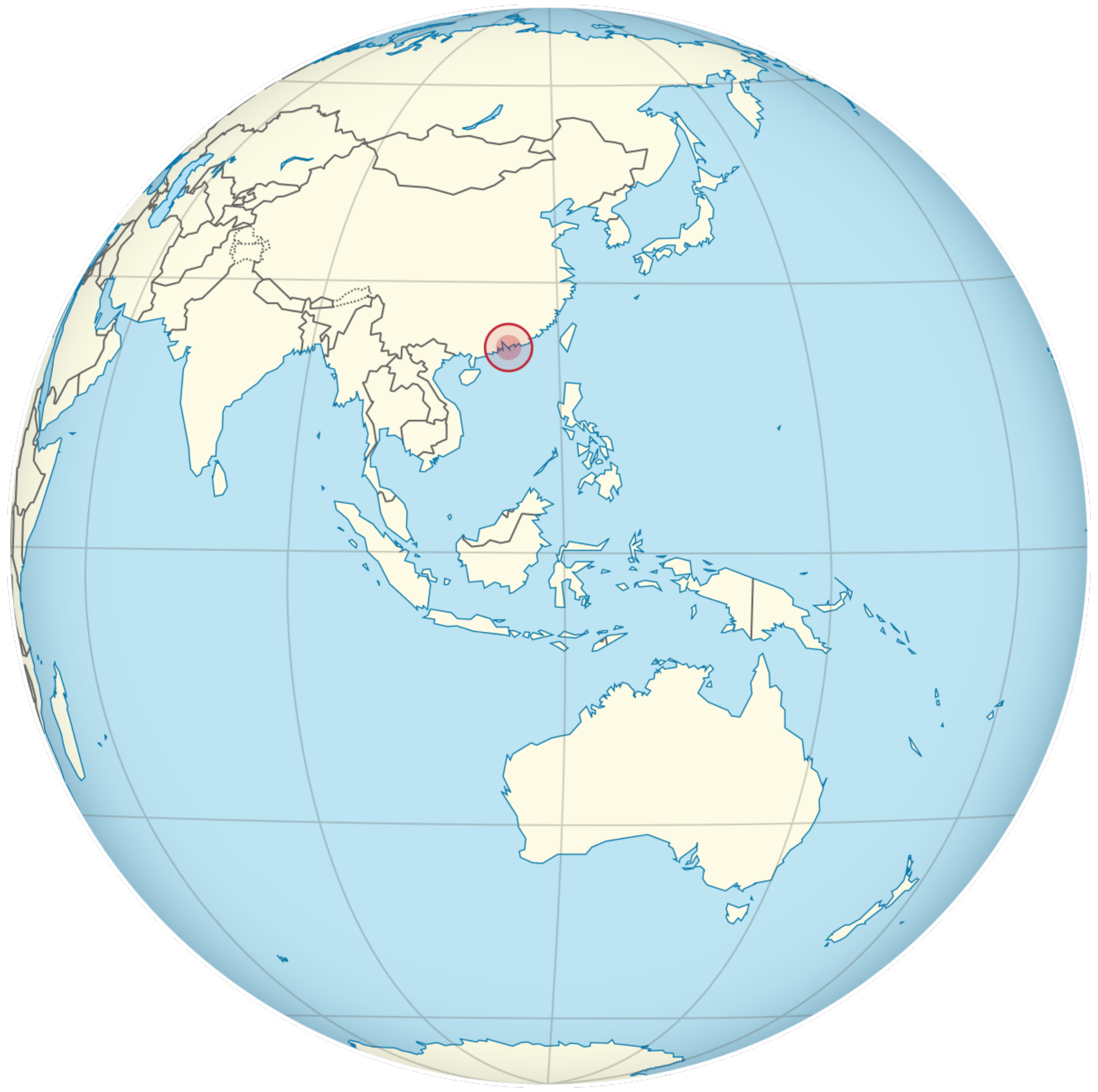
A coffee mug is a donut.

Homeomorphism

Luckily, we can draw a map of Hong Kong.

Claim: 地球上的香港 is homeomorphic to 地图上的香港。

k-dimensional topological manifold（拓扑流形）: “locally homeomorphic to $\mathbb{R}^k$”.



References

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